数理解析研究所講究録
467巻 1982年 74-82

Classification of T^2-bundles over T^2 (summary)

by Koichi Sakamoto and Shinji Fukuhara

(坂本幸一，福原真二)

§1. Notations and definitions

Given $A, B \in GL(2,\mathbb{Z})$ such that $AB = BA$, and $m, n \in \mathbb{Z}$, we construct a T^2-bundle over T^2 denoted by

$$\pi : M(A,B;m,n) \to S,$$

as follows.

Denote by $\begin{bmatrix} x \\ y \end{bmatrix}$ the point of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ corresponding to $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. Let $F = T^2$, $S = T^2$, and we define

$$M(A,B;0,0) = F \times \mathbb{R}^2/\sim$$

where

$$\left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{pmatrix} x+1 \\ y \end{pmatrix}\right) \sim \left(\begin{bmatrix} A\begin{pmatrix} s \\ t \end{pmatrix} \end{bmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right),$$

and

$$\left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{pmatrix} x \\ y+1 \end{pmatrix}\right) \sim \left(\begin{bmatrix} B\begin{pmatrix} s \\ t \end{pmatrix} \end{bmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right).$$

Denote the point of $M_0 = M(A,B;0,0)$ which corresponds to $\left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right)$, by $\left[\begin{pmatrix} s \\ t \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right]$ or $\begin{bmatrix} s,x \\ t,y \end{bmatrix}$.

Then $\pi : M_0 \to S$ is a T^2-bundle over T^2, where π is defined by $\pi\begin{bmatrix} s,x \\ t,y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. Let D be a small disk in S centered at $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ with radius ε, and let

$$M(A,B;m,n) = (M_0 - \pi^{-1}(\mathrm{Int}\ D)) \cup (F \times D)$$

where $F \times \partial D$ is attached to $\pi^{-1}(\partial D)$ by the homeomorphism $h : \pi^{-1}(\partial D) \to F \times \partial D$:

$$h\left[\begin{pmatrix} s \\ t \end{pmatrix}, \varepsilon(\theta) \right] = \left(\left[\begin{pmatrix} s \\ t \end{pmatrix} + (\theta/2\pi) \begin{pmatrix} m \\ n \end{pmatrix} \right], \left[\varepsilon(\theta) \right] \right)$$

where $\varepsilon(\theta) = \begin{pmatrix} 1/2+\varepsilon\cos\theta \\ 1/2+\varepsilon\sin\theta \end{pmatrix}$.

Define the map $\pi : M(A,B;m,n) \to S$

$$\pi \begin{bmatrix} s,x \\ t,y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{if} \quad \begin{bmatrix} x \\ y \end{bmatrix} \notin D, \quad \text{and}$$

$$\pi \left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{if} \quad \begin{bmatrix} x \\ y \end{bmatrix} \in D.$$

Then this is a T^2-bundle over T^2.

Every T^2-bundle over T^2 is isomorphic to this form, where the pair (A,B) represents the monodromy, and the pair (m,n) represents the obstruction for constructing a cross-section. Corresponding to a T^2-bundle over T^2, $\pi : M \to S$, there is an exact sequence

$$1 \to \pi_1 F \to \pi_1 M \to \pi_1 S \to 1$$

where F is a fiber. We call this the associated exact sequence.

§2. Fundamental lemmas.

Proposition 1. $H_1(M(A,B;m,n))$ is isomorphic to $\mathbb{Z}^2 \oplus (\mathbb{Z}^2/K)$, where K is the subgroup of $\mathbb{Z}^2$ generated by

$\binom{m}{n}$ and the column vectors of $A-E$ and $B-E$ (E stands for $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$).

In the following proposition, we study about typical bundle isomorphisms.

Proposition 2. Let $A, B, A', B' \in GL(2,\mathbb{Z})$ such that $AB = BA$ and $A'B' = B'A'$. Let $\alpha, \beta, \sigma, \tau$ and $\alpha', \beta', \sigma', \tau'$ are canonical generators of $\pi_1 M$ and $\pi_1 M'$ respectively, where $M = M(A,B;m,n)$ and $M' = M(A',B';m',n')$.

(1) Assume $A' = A^p B^r$, $B' = A^q B^s$ and $\binom{m'}{n'} = \delta \binom{m}{n}$ for some $P = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in GL(2,\mathbb{Z})$ where $\delta = ps - qr = \pm 1$. Then there is a bundle isomorphism $f : M' \to M$ such that

$$f_{\#}(\sigma') = \sigma, \quad f_{\#}(\tau') = \tau \quad \text{and}$$

$$f_{\#}(\alpha') = \alpha^p \beta^r, \quad f_{\#}(\beta') = \alpha^q \beta^s$$

where $f : S' \to S$ is a corresponding homeomorphism between base spaces and $\alpha = \pi_{\#}(\alpha)$ etc.

(2) Assume $A' = P^{-1}AP$, $B = P^{-1}BP$ and $\binom{m}{n} = P\binom{m'}{n'}$ for some $P = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in GL(2,\mathbb{Z})$. then there is a bundle isomorphism $f : M' \to M$ such that

$$f_{\#}(\alpha') = \alpha, \quad f_{\#}(\beta') = \beta \quad \text{and}$$

$$f_{\#}(\sigma') = \sigma^p\tau^r, \quad f_{\#}(\tau') = \sigma^q\tau^s.$$

(3) Assume $A' = A$, $B' = B$ and $\begin{pmatrix} m' \\ n' \end{pmatrix} - \begin{pmatrix} m \\ n \end{pmatrix}$ $= (A-E)\begin{pmatrix} p \\ q \end{pmatrix} + (B-E)\begin{pmatrix} k \\ l \end{pmatrix}$ for some $p, q, k, l \in \mathbb{Z}$. Then there is a bundle isomorphism $f : M' \to M$ such that

$$f_{\#}(\alpha') = \sigma^{k'}\tau^{l'}\alpha, \quad f_{\#}(\beta') = \sigma^{p'}\tau^{q'}\beta \quad \text{and}$$

$$f_{\#}(\sigma') = \sigma, \quad f_{\#}(\tau') = \tau,$$

where $\begin{pmatrix} k' \\ l' \end{pmatrix} = B\begin{pmatrix} k \\ l \end{pmatrix}$ and $\begin{pmatrix} p' \\ q' \end{pmatrix} = A\begin{pmatrix} p \\ q \end{pmatrix}$.

Remark. The last result (3) of the above proposition corresponds to the fact that the obvstruction class to construcing a cross section lies in $H^2(S, \widetilde{\pi_1}(F))$ ($\widetilde{\pi_1}(F)$ is the locally constant sheaf whose stalk at $x \in S$ is naturally isomorphic to $\pi_1 F_x$, where $F_x = \pi^{-1}(x)$), and that $H^2(S, \widetilde{\pi_1}(F))$ is isomorphic to the quotient group $\mathbb{Z}^2/\langle A-E, B-E\rangle$ where $\langle A-E, B-E\rangle$ is the subgroup generatedby the column vector of $A-E$ and $B-E$.

§3. Main results.

The problem of bundle isomorphisms is reduces to the group theory of the associated exact sequences by the following theorem.

Theorem 1. Let $\pi : M \to S$ and $\pi' : M' \to S'$ be T^2-bundles over T^2. Then the following statements are equivalent.

1) They are bundle isomorphic to each other.

2) The associated exact sequences of them are isomorphic to each other, that is, there exist isomorphism of groups $\psi : \pi_1 M' \to \pi_1 M$ and $\overline{\psi} : \pi_1 S' \to \pi_1 S$ such that $\pi_{\#} \circ \psi = \overline{\psi} \circ (\pi')_{\#}$.

Corollary. Two fibrations $M(A,B;m,n)$ and $M(A',B';m',n')$ are isomorphic if and only if there exist $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ and $P \in GL(2,\mathbb{Z})$ as follows:

$$A^p B^r = PA'P^{-1}, \quad A^q B^s = PB'P^{-1} \quad \text{and}$$

$$P\begin{pmatrix} m' \\ n' \end{pmatrix} - \begin{pmatrix} m \\ n \end{pmatrix} \in \langle A-E,\ B-E \rangle,$$

where $\langle A-E, B-E \rangle$ is the subgroup of $\mathbb{Z}^2$ generated by the column vectors of $A-E$ and $B-E$.

Theorem 2. Let $\pi : M \to S$ and $\pi' : M' \to S'$ be T^2-bundles over T^2.

(1) rank $(H_1 M) = 4$ if and only if $M = M(E,E;0,0)$, which is a 4-dimensional torus.

(2) Assume rank $(H_1 M) \neq 3$. Then the above fibrations are

isomorphic if and only if $\pi_1 M$ and $\pi_1 M'$ are isomorphic.

Any fibration has a simple expression as follows:

Theorem 3. Any T^2-bundle over T^2 is isomorphic to one of the following types:

$M(A,B;m,n)$ where $B = \pm E$.

Furthermore, we may assume that A satisfies the following conditions:

(1) if $\det A = -1$, $\operatorname{trace} A \geq 0$,

(2) if $\det A = 1$ and $B = -E$, $\operatorname{trace} A \geq 2$ and $A = E$,

Remark. Under the above assumption (2), $B = E$ if and only if the subgroup of $GL(2,\mathbb{Z})$ generated by A and B is a cyclic group. The conjugacy class of this group in $GL(2,\mathbb{Z})$ is an invariant of the associated exact sequence. In fact, if $\rho : \pi_1 S \to \operatorname{Aut}(\pi_1 F)$ is the homomorphism defined by

$$\rho(\pi_{\#}(x))(y) = x^{-1}yx \quad (x \in \pi_1 M,\ y \in \pi_1 F \subset \pi_1 M),$$

then $\operatorname{Im} \rho$ is mapped onto the above group by a global isomorphism from $\mathrm{Aut}(\pi_1 F)$ to $GL(2,\mathbb{Z})$.

Theorem 4. Assume $M = M(A,B;m,n)$ and $M' = M(A',B';m',n')$ satisfy the condition of Theorem 3. Denote by $\langle A-E\rangle$ the subgroup of of $\mathbb{Z}^2$ generated by the vectors of $A-E$, and similarly for $\langle A-E, 2E\rangle$.

(0) If M and M' are bundle isomorphic to each other, then $B = B'$.

(1) Assume, $B = B' = E$. Then M is bundle isomorphic to M', if and only if there exists a matrix $P \in GL(2,\mathbb{Z})$ such that

i) $PA'P^{-1} = A$ or $PA'P^{-1} = A^{-1}$ and

ii) $\begin{pmatrix} m \\ n \end{pmatrix} - P\begin{pmatrix} m' \\ n' \end{pmatrix} \in \langle A-E\rangle$.

(2) Assume, $B = B' = -E$. Then M is bundle isomorphic to M', if and only if there exists a matrix $P \in GL(2,\mathbb{Z})$ such that

i) $PA'P^{-1} = \pm A$ or $PA'P^{-1} = \pm A^{-1}$ and

ii) $\begin{pmatrix} m \\ n \end{pmatrix} - P\begin{pmatrix} m' \\ n' \end{pmatrix} \in \langle A-E, 2E\rangle$.

§4. Homeomorphism types.

Let $\pi : M \to S$ be a T^2-bundle over T^2. If rank $(H_1M) \neq 3$, the bundle isomorphism type is determined by $\pi_1 M$ (Theorem 2).

Now we consider the case when rank $(H_1M) = 3$. According to proposition 1, rank $(H_1(M(A,B;m,n)) = 3$ if and only if the rank of the 2×5 matrix $\left(A-E,\ B-E,\ \begin{smallmatrix} m \\ n \end{smallmatrix}\right)$ is equal to 1. Hence in view of Theorem 3, M is isomorphic to one of the following forms:

1) $M\left(\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, E; m, 0\right)$ $(k \geq 0)$

2) $M\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E; 0, n\right)$ or

3) $M\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, E; m, -m\right)$.

Furthermore, we have:

Proposition 3. If rank $(H_1M) = 3$, M is homeomorphic to one and only one of the following forms:

1) $M\left(\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, E; o, o\right)$ $(d > 0)$

2) $M\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E; 0, n\right)$ $(n = 0$ or $1)$

Corollary. Let $\pi : M \to S$ and $\pi' : M' \to S'$ be T^2-bundle over T^2. Assume that M and M' are both orientable or both non-orientable, and rank H_1M = rank $H_1M' = 3$. Then M is homeomorphic to M' if and only if $H_1M \cong H_1M'$.

Remark. The orientablity of M is an invariant of $\pi_1 M$. In fact, let $\rho : H_1M \to \mathrm{Aut}([\pi_1, \pi_1])$, where $[\pi_1, \pi_1]$ is the commutator subgruop of $\pi_1 M$ and ρ is the homomorphism which is defined similarly to the remark to Theorem 3.
When rank $H_1M = 3$, by the above proposition, we see that ρ is a trivial map if and only if M is orientable.

This remark and Theorem 2 imply:

Theorem 5. Let $\pi : M \to S$ and $\pi' : M' \to S'$ be T^2-bndles over T^2. Then M ia homeomorphic to M' if and only if $\pi_1 M$ is isomorphic to $\pi_1 M'$.